NOTE ON THE CLASSIFICATION OF SMOOTH (k, p)-COVERS OF THE PLANE OF GEOMETRIC GENUS THREE

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ABSTRACT. In this note we classify smooth surfaces with geometric genus equal to three admitting an action of a group G isomorphic to $\mathbb{Z}_p^k$, with p being a prime number, such that the quotient is a plane.

1. INTRODUCTION

We classify all smooth surfaces with geometric genus equal to three admitting an action of a group G isomorphic to $\mathbb{Z}_p^k$, with p being a prime number, such that the quotient is a plane. We call such surfaces smooth (k, p)-covers of the plane of geometric genus three.

Following [8], then each smooth (k, p)-covers of $\mathbb{P}^2$ is built by a set of algebraic data that uniquely identifies the cover. Such algebraic data consists of the components $\{D_g\}_{g\in\mathbb{Z}_p^k}$ of the branch locus of the quotient map $X \to \mathbb{P}^2$ and a collection of line bundles $\{\mathcal{L}_\chi\}_{\chi\in\mathbb{Z}_p^k}$ of $\mathbb{P}^2$ fitting in a set of linear equations, those of Theorem 2.2.

Hence a classification of smooth (k, p)-covers means that we determine from the linear equations of Theorem 2.2 all the possibilities of the algebraic data $\{D_g\}_{g\in\mathbb{Z}_p^k}$ and $\{\mathcal{L}_\chi\}_{\chi\in\mathbb{Z}_p^k}$.
Following the same technique in [4], it is possible to translate those algebraic data to define any (k, p)-cover by equations in suitable weighted projective spaces.

A classification for $p = 2$ and any natural integer k is already available in [4], hence, whenever not specified, we assume in this note $p \geq 3$. The main argument of [4] for which the classification has been accessible was the formula in [4, Theorem 1.11]. This formula computes the numerical class of all divisors D_g by the characteristic line bundles $\mathcal{L}_\chi$, if the Galois group is of the form $\mathbb{Z}_2^k$. Hence the authors first compute the possible $\mathcal{L}_\chi$, that is easy, and then deduce from it the class of each divisor D_g. However, formula [4, Theorem 1.11] is not true for a general abelian group, since different numerical class of divisors may give the same characteristic sheaves L_χ, see [4, Example 1.7].
In this note we deduce a general formula (2.6) for an abelian group, and then we specialize it to the group $G = \mathbb{Z}_p^k$ (see (2.8)). We observe that this is the formula of [4, Theorem 1.11] whenever $p = 2$. Although such formula can not permit to

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compute the numerical class of each divisor D_g in function of the sheaves $\mathcal{L}_\chi$, we get from it some crucial constraints on the natural integers k and p for which a classification is feasible (see in order Proposition 4.5, Proposition 4.6 and Theorem 4.14).

We find four families of smooth $(k,p)-$planes of geometric genus three, that together with the eleven ones obtained in [4], provides a complete classification of smooth (k,p)-planes of geometric genus three.

The main result of this note is the following Theorem 4.22.

Theorem. *Let $p \geq 3$ be a prime number. All smooth (k,p)-covers X of the plane with geometric genus 3 have $p = 3$ and are regular surfaces with ample canonical class.*

They form 4 unirational families, that we labeled as A1, A2, B3, C2, in a way that A1 is a family of triple planes, A2 and C2 are two families of bi-triple planes and B3 is a family of $(3,3)$-planes.

The degree of the canonical map φ_{K_X} is constant in each family.

We summarize in the following table the modular dimension (the dimension of its image in the Gieseker moduli space of the surfaces of general type) of each family, and the values of K_X^2 and $\deg \varphi_{K_X}$ of each surface in the family: The

Family	*A1*	*A2*	*B3*	*C2*
mod. dim.	*19*	*7*	*3*	*7*
K_X^2	*3*	*9*	*3*	*9*
$\deg \varphi_{K_X}$	*3*	*9*	*3*	*3*

canonical map is a morphism of degree K_X^2 on $\mathbb{P}^2$ unless X of type $C2$, in which case the canonical map is a rational map of degree $K_X^2 - 3 \cdot 2 = 3$ undefined at 3 points.

The building algebraic data $\{D_g\}_{g\in\mathbb{Z}_p^k}$, $\{\mathcal{L}_\chi\}_{\chi\in\mathbb{Z}_p^k}$ can be found in propositions 4.6, 4.17, and 4.19.

Notation. The notation is the same as [4, Chapter 1]. Thus, a Galois cover is a finite morphism $\pi\colon X \to Y$ among algebraic varieties with the property that there is a subgroup G of $\mathrm{Aut}(X)$ such that π factors as the composition of the quotient map $X \to X/G$ with an isomorphism $X/G \cong Y$. We will always assume Y to be irreducible, whereas we find it convenient for the general theory of Galois covers not to do any analogous assumption for X. The finite group G is the Galois group of π.

An abelian cover is a Galois cover whose Galois group is an abelian group. A (k,p)-cover is an abelian cover whose Galois group is isomorphic to $\mathbb{Z}_p^k$. A (k,p)-plane is a (k,p)-cover of $\mathbb{P}^2$.

2. Abelian covers

In this section we collect some known results on abelian covers. The novelty is formula in Theorem 2.8, which is a natural generalization of that present in

[4, Theorem 1.11] to any abelian group. We are going to use the same notation of [4] and we remind to the reader only the most important facts on abelian covers; for more details see [8] and [4].

Definition 2.1. Let $\pi\colon X \to Y$ be an abelian cover with Galois group G, Y smooth and X normal. Fix an element $g \in G$ and a character $\chi \in G^*$. Let $o(g)$ be the order of g. Then there exists a unique integer $0 \leq r_g^\chi \leq o(g) - 1$ such that

$$\chi(g) = e^{r_g^\chi \cdot \frac{2\pi i}{o(g)}}.$$

Given a further character $\chi' \in G^*$ we set moreover

$$\varepsilon^g_{\chi,\chi'} = \begin{cases} 1 & \text{if } r_g^\chi + r_g^{\chi'} \geq o(g) \\ 0 & \text{else} \end{cases}.$$

Theorem 2.2 ([8, Theorem 2.1 and Corollary 3.1])**.** *Let $\pi\colon X \to Y$ be an abelian cover with Galois group G, Y smooth and X normal.*

Then for all $\chi, \chi' \in G^*$

$$\mathcal{L}_\chi \otimes \mathcal{L}_{\chi'} \cong \mathcal{L}_{\chi\cdot\chi'} \otimes \mathcal{O}_X\left(\sum_{g\in G} \varepsilon^g_{\chi,\chi'} \cdot D_g\right). \tag{2.1}$$

Conversely, given an abelian group G and a smooth irreducible variety Y assume that we have

a line bundle $\mathcal{L}_\chi$ on Y for each character $\chi \in G^*$ and
an effective divisor D_g for all $g \in G$

satisfying (2.1), and with the property that the divisor $D = \sum D_g$ is reduced.

Then there is a unique Galois cover $\pi\colon X \to Y$ whose Galois group is G, and whose building data are the $\mathcal{L}_\chi$ and the D_g, such that X is normal.

Equation (2.1) shows that the divisors D_g determine the line bundles $\mathcal{L}_\chi$ up to torsion as follows.

Definition 2.3. For all χ set $L_\chi \in \mathrm{Pic}(Y) = \mathrm{Div}(Y)/\sim$ for the divisor class of the invertible sheaf $\mathcal{L}_\chi$. We use the additive notation for the torsion product in $\mathrm{Pic}(Y)$.

Corollary 2.4 (see [8, Proposition 2.1])**.**

$$o(\chi)L_\chi \equiv \sum_{g\in G} \frac{o(\chi) r_g^\chi}{o(g)} D_g.$$

In particular

$$L_\chi \equiv_{num} \sum_{g\in G} \frac{r_g^\chi}{o(g)} D_g.$$

In particular, if $\mathrm{Pic}(Y)$ is torsion free (for example if Y is rational) then the divisors do determine the line bundles.

In the next sections we are going to walk in the opposite direction: first we look for the "good" possible $\mathcal{L}_\chi$ and then we find suitable divisors D_g realizing them.

Of course the divisors will be free to move in their linear equivalence class. We find it important to notice that for general G the line bundles $\mathcal{L}_\chi$ do not determine even the linear equivalence class of the divisors D_g. In fact this fails already for cyclic groups of order 5 or more, see for instance the [4, Example 1.7].

In contrast, we show in the forthcoming Theorem 2.9 that when $G \cong \mathbb{Z}_p^k$ the L_χ determine the linear equivalence class of the divisors $\sum_{h\in\langle g\rangle} D_h$, $g \in G$, up to torsion. In particular, for $p = 2$ we get the already proved [4, Theorem 1.11], for which any D_g is uniquely determined by the L_χ.

We first need some lemmas for general abelian covers.

Definition 2.5. The natural isomorphism $G \to G^{**}$ allows each g in G to be considered as a character of G^*, which we will also denote by g, by setting

$$g(\chi) = \chi(g).$$

Then $\ker g$ is the subgroup of G^* of the characters χ such that $\chi(g) = 1$. In other words

$$\chi \in \ker g \Leftrightarrow g \in \ker \chi.$$

Let $\mathcal{H}$ be a subgroup of G^*, possibly of the form $\ker g$. For all $g \in G$ we will denote by $g_{|\mathcal{H}}$ the element of $\mathcal{H}^*$ obtained restricting g to $\mathcal{H}$.

Lemma 2.6. [4, Lemma 2.9] *For all $g \in G$, for each subgroup $\mathcal{H}$ of G^*,*

$$\sum_{\chi\in\mathcal{H}} r_g^\chi = \frac{|\mathcal{H}|}{2} o(g)\left(1 - \frac{1}{o(g_{|\mathcal{H}})}\right) \tag{2.2}$$

In particular

$$\sum_{\chi\in G^*} r_g^\chi = \frac{|G|}{2}\left(o(g) - 1\right). \tag{2.3}$$

It follows that

Proposition 2.7. [4, Prop. 2.10]

$$\sum_{\chi\in G^*} L_\chi \equiv_{num} \frac{|G|}{2} \sum_{g\in G}\left(1 - \frac{1}{o(g)}\right) D_g. \tag{2.4}$$

Moreover, for every $g \in G$,

$$\sum_{\chi\in\ker g} L_\chi \equiv_{num} \frac{|G|}{2o(g)} \sum_{h\in G}\left(1 - \frac{1}{o(h_{|\ker g})}\right) D_h. \tag{2.5}$$

The main result of this chapter is

Theorem 2.8. *Let $\pi\colon X\to Y$ be a G-cover, Y smooth and X normal, with set of data $\mathcal{L}_\chi$, D_g. Then for all subgroup $\mathcal{H}$ of G^**

$$\frac{|G|}{2}\sum_{g\in G}\left(\frac{1}{o(g_{|\mathcal{H}})}-\frac{1}{o(g)}\right)D_g \equiv_{num} \sum_{\chi\notin\mathcal{H}} L_\chi - ([G^*:\mathcal{H}]-1)\sum_{\chi\in\mathcal{H}} L_\chi. \tag{2.6}$$

Proof. Consider a subgroup $\mathcal{H}$ of G^*. Then by (2.2), one has

$$\begin{aligned}\sum_{\chi\in\mathcal{H}} L_\chi \equiv_{num} \sum_{\chi\in\mathcal{H}}\left(\sum_{g\in G}\frac{r_g^\chi}{o(g)}D_g\right) &\equiv_{num} \sum_{g\in G}\frac{1}{o(g)}\left(\sum_{\chi\in\mathcal{H}} r_g^\chi\right)D_g \\ &\equiv_{num} \frac{|\mathcal{H}|}{2}\sum_{g\in G}\left(1-\frac{1}{o(g_{|\mathcal{H}})}\right)D_g.\end{aligned} \tag{2.7}$$

Using (2.7) and (2.4) we obtain

$$\begin{aligned}\frac{|G|}{2}\sum_{g\in G}\left(\frac{1}{o(g_{|\mathcal{H}})}-\frac{1}{o(g)}\right)D_g &\equiv_{num} \frac{|G|}{2}\sum_{g\in G}\left(1-\frac{1}{o(g)}\right)D_g - \frac{|G|}{2}\sum_{g\in G}\left(1-\frac{1}{o(g_{|\mathcal{H}})}\right)D_g \\ &\equiv_{num} \sum_{\chi\in G^*} L_\chi - [G^*:\mathcal{H}]\sum_{\chi\in\mathcal{H}} L_\chi \\ &\equiv_{num} \sum_{\chi\notin\mathcal{H}} L_\chi - ([G^*:\mathcal{H}]-1)\sum_{\chi\in\mathcal{H}} L_\chi.\end{aligned}$$

□

Corollary 2.9. *Let $\pi\colon X\to Y$ be a $\mathbb{Z}_p^k-$cover, Y smooth and X normal, with set of data $\mathcal{L}_\chi$, D_g. Then for all subspace $\mathcal{H}$ of G^* of codimension t, we have*

$$\frac{p^{k-1}(p-1)}{2}\sum_{h\in\mathcal{H}^\perp} D_h \equiv_{num} \sum_{\chi\notin\mathcal{H}} L_\chi - (p^t-1)\sum_{\chi\in\mathcal{H}} L_\chi.$$

In particular, given $g\in G$, $g\neq 0$, and setted $\mathcal{H}:=\ker g$, then the above formula becomes

$$\frac{p^{k-1}(p-1)}{2}\sum_{h\in\langle g\rangle} D_h \equiv_{num} \sum_{\chi\notin\ker g} L_\chi - (p-1)\sum_{\chi\in\ker g} L_\chi. \tag{2.8}$$

Proof. We notice that for all h in G, then $o(h_{|\mathcal{H}})$ equals 1 if $h\in\mathcal{H}^\perp$, and is equal to p otherwise. Hence the sum of the left member of (2.6) is on $\mathcal{H}^\perp$. Since $\mathcal{H}$ is a subspace of codimension t, then it has order p^{k-t}, and so its index in G^* amounts to $[G^*:\mathcal{H}]=p^t$.

The second equation of the statement follows once one observes that if $\mathcal{H}=\ker g$, then $\mathcal{H}^\perp=\langle g\rangle$. Indeed, the inclusion $\langle g\rangle\subseteq\mathcal{H}^\perp$ is straightforward, whilst the other direction holds directly by using that each character defined over a proper subgroup can be extended to a non-trivial character of the entire group. □

Now we give a formula useful to determine the characteristic $e(X)$ of a smooth abelian cover with Galois group G of a smooth algebraic surface Y.

Proposition 2.10. *Let $\pi\colon X \to Y$ be a smooth abelian cover with Galois group G and Y smooth. Let $D = \sum_g D_g$ be the branch locus of π. Then*

$$e(X) = |G|\left(e(Y) - \sum_g \left(1 - \frac{1}{o(g)}\right) e(D_g) + \frac{1}{2}\sum_{g\neq h}\left(1 - \frac{1}{o(g)}\right)\left(1 - \frac{1}{o(h)}\right) D_g \cdot D_h\right). \tag{2.9}$$

Proof. $e(X) = n_0 - n_1 + n_2 - n_3 + n_4$ where n_i is the number of cells of dimension i. We can choose a particular cellular decomposition on Y such that

- The intersection points between D_g and D_h are 0-cells of the decomposition;
- The cellular decomposition of Y induces a cellular decomposition on each D_g;
- The cells are contained on some D_g or they do not touch the branch locus D.

Such particular decomposition on Y induces via pre-image a cellular decomposition on X. If the covering π is not ramified, then any cell on Y is counted exactly $|G|$-times on X, hence $e(X) = |G|e(Y)$.

Instead, if the covering is ramified, then we have to add a correction term. Any 2-cell is counted $|G|$ times except for the components D_g of the branch locus, that is counted $\frac{|G|}{|\operatorname{Stab}(D_g)|} = \frac{|G|}{o(g)}$-times. Hence to $|G|e(Y)$ we have to subtract the correction term

$$\sum_g \left(|G| - \frac{|G|}{o(g)}\right) e(D_g).$$

However, there is another correction term to consider, since the 0-cells are counted $|G|$-times except for the intersection points p of $D_g \cap D_h$, $g \neq h$. Indeed, they are counted $\frac{|G|}{|\operatorname{Stab}(p)|} = \frac{|G|}{|\langle g,h\rangle|}$-times. Since the covering is smooth, then [8, Proposition 3.1] implies $\langle g, h\rangle = \langle g\rangle \oplus \langle h\rangle$, and so $|\langle g, h\rangle| = o(g)o(h)$.
Hence the last correction term to add is

$$\frac{1}{2}\sum_{g\neq h}\left(\left(|G| - \frac{|G|}{o(g)}\right) + \left(|G| - \frac{|G|}{o(h)}\right) - \left(|G| - \frac{|G|}{o(g)o(h)}\right)\right) D_g \cdot D_h.$$

□

We finish the section with a remark on the Galois covers of $Y = \mathbb{P}^2$.

Remark 2.11. Any Galois cover $\pi\colon S \to \mathbb{P}^2$ have irregularity zero.

Indeed, from the Leray spectral sequence, then

$$H^1(S, \mathcal{O}_S) \cong H^1(\mathbb{P}^2, \pi_*\mathcal{O}_S) \cong \bigoplus_\chi H^1(\mathbb{P}^2, \mathcal{L}_\chi^{-1}).$$

Since every line bundle of $\mathbb{P}^2$ has trivial first cohomology group, then $h^1(S, \mathcal{O}_S) = 0$.

3. The canonical system of an abelian cover

A canonical divisor K_X on a normal variety X is a Weil divisor, the closure of a canonical divisor of the smooth part of X (see [7, (1.5)]).

If $\pi\colon X \to Y$ is a $G-$cover, then G acts on $\pi_*(\mathcal{O}_X(K_X))$ inducing a decomposition on it in eigenspaces

$$\pi_*(\mathcal{O}_X(K_X)) = \bigoplus_{\chi\in G^*} \pi_*(\mathcal{O}_X(K_X))^{(\chi)}$$

Theorem 3.1 ([1, Proposition 2.4], see also [8, Proposition 4.1, c) for the case when X is smooth])**.** *Let $\pi\colon X \to Y$ be an abelian cover, with X normal and Y smooth, whose building data are $\mathcal{L}_\chi$ and D_g. Then*

$$(\pi_*\mathcal{O}_X(K_X))^{(\chi)} \cong \mathcal{O}_Y(K_Y)\otimes\mathcal{L}_{\chi^{-1}}. \tag{3.1}$$

Consider a global section $\sigma \in H^0(\mathcal{O}_Y(K_Y)\otimes\mathcal{L}_{\chi^{-1}})$, and let $(\sigma)\in \mathrm{Div}(Y)$ be the induced effective divisor. By (3.1) σ determines an element of $H^0(\pi_*\mathcal{O}_X(K_X)) \cong H^0(\mathcal{O}_X(K_X))$, whose divisor is, by the proof of [1, Proposition 2.4] (compare also [6, Section 3.4]),

$$\pi^*(\sigma) + \sum_g (o(g) - r_g^{\chi^{-1}} - 1)R_g. \tag{3.2}$$

4. Smooth (k-p) planes with p_g=3

Definition 4.1. A smooth (k,p)-plane is a (k,p)-cover $\pi\colon X\to\mathbb{P}^2$ such that all D_h are smooth, each two of them intersect transversally, and no point in $\mathbb{P}^2$ belongs to three of them. Furthermore we require that

$$\langle g\rangle \cap \langle h\rangle = \{0\}. \tag{4.1}$$

for any pairwise g and h with $D_h, D_g \neq 0$, $h\neq g$.

Notice that the branch divisor $D=\sum D_g$ is a smooth normal crossing divisor.

The assumption ensures the smoothness of X.

Proposition 4.2. *Let $\pi\colon X\to\mathbb{P}^2$ be a smooth (k,p)-plane. Then X is smooth.*

Proof. This is a special case of [8, Proposition 3.1]. □

Notation 4.3. It is convenient to consider G and G^* as vector spaces over the field with p elements. We are thus going to switch to the additive notation, so for example the sheaf $\mathcal{L}_1$ will be $\mathcal{L}_0$ from now on, and for each character χ we will write $-\chi$ for the character that was called χ^{-1} in the previous section.

Denote by $e_1,\cdots,e_k$ the standard basis of $G=\mathbb{Z}_p^k$ and by $\epsilon_1,\dots,\epsilon_k$ the dual basis of G^*.

To every (k,p)-plane $\pi\colon X\to\mathbb{P}^2$ we consider its building data L_χ, D_g and the numbers

$$d_g := \deg D_g, \qquad l_\chi := \deg L_\chi.$$

Note that $d_0 = l_0 = 0$.

Note moreover that since $G = \mathbb{Z}_p^k$, for each $\chi \in G^*$, $(p-1)\chi = -\chi$. We will use this often in the following computations.

Definition 4.4. We will say that a smooth (k,p)-plane with $p_g = 3$ is

of type A if $l_{\epsilon_1} = 4$, $l_\chi \in \{1,2\}$ for all $\chi \neq \epsilon_1$

of type B if $l_{\chi_1} = l_{\chi_2} = l_{\chi_3} = 3$, $l_\chi \in \{1,2\}$ for all $\chi \neq \chi_1, \chi_2, \chi_3$.

For a smooth (k,p)-plane $\pi\colon X \to \mathbb{P}^2$

$$p_g(X) = h^0(\mathcal{O}_X(K_X)) = h^0(\pi_*(\mathcal{O}_X(K_X))) = \sum_{\chi \in G^*} h^0(\mathcal{O}_{\mathbb{P}^2}(l_\chi - 3)), \tag{4.2}$$

so in all cases of Definition 4.4 we obtain $p_g(X) = 3$. Conversely

Proposition 4.5. *Up to automorphisms of G every smooth (k,p)-plane with $p_g(X) = 3$ falls in one of the two cases in Definition 4.4.*

Proof. Since X is connected, for all $\chi \neq 0$, $H^0(\mathcal{L}_\chi^{-1}) = 0$ and thus $l_\chi > 0$.

By (4.2) $l_\chi \leq 4$ and either there is only one χ with $l_\chi \geq 3$, in which case $l_\chi = 4$, or there are three χ with $l_\chi \geq 3$, all with $l_\chi = 3$.

Using an automorphism of G, we can reduce the former case to "type A". $\square$

We can now classify the (k,p)-planes with $p_g(X) = 3$ considering separately the cases in Definition 4.4. We remember that we assume $p \geq 3$ since a classification for $p = 2$ is already available, see [4].

For type A we obtain a special case of the situation classified in [3, Theorem 1.1].

Proposition 4.6. *The generic (k,p)-planes of "type A" form 2 families, the first with $(k,p) = (1,3)$ and the second with $(k,p) = (2,3)$.*

In both cases π is the canonical map of X, $|K_X| = |\pi^*\mathcal{O}_{\mathbb{P}^2}(1)|$ is base point free and

$$l_0 = 0 \qquad l_{\epsilon_1} = 4 \qquad l_\chi = 2 \text{ for all remaining } \chi$$

$$d_g = 2 \cdot 3^{2-k} \quad \text{for} \quad \epsilon_1(g) = 2, \qquad d_g = 0 \text{ otherwise.}$$

Proof. By (2.1), for all $\chi \in G^*$, we have

$$l_\chi + l_{(p-1)\chi+\epsilon_1} = l_{\epsilon_1} + \sum_{g \in G} \varepsilon^g_{\chi,(p-1)\chi+\epsilon_1} d_g \geq l_{\epsilon_1} = 4.$$

Since for χ distinct to $0, \epsilon_1$ we have $l_\chi \leq 2$, it follows $l_\chi = 2$.

By Theorem 2.9, for all $g \notin \ker \epsilon_1$, then

$$\begin{aligned}\frac{p^{k-1}(p-1)}{2} \sum_{h \in \langle g \rangle} d_h &= \sum_{\chi \notin \ker g} l_\chi - (p-1) \cdot \sum_{\chi \in \ker g} l_\chi \\ &= \left(4 + (p^{k-1}(p-1) - 1) \cdot 2\right) - (p-1)\left(0 + (p^{k-1} - 1) \cdot 2\right) \\ &= 2 + 2(p-1) = 2p\end{aligned}$$

Therefore

$$\sum_{g\in\langle h\rangle} d_g = \frac{2}{p^{k-1}(p-1)}2p = \frac{4p^{2-k}}{p-1}, \tag{4.3}$$

that is an integer for $p \in \{2,3,5\}$.

Note also that for all $g \in \ker \epsilon_1$, we have

(4.4)

$$\frac{p^{k-1}(p-1)}{2}\sum_{h\in\langle g\rangle} d_h = \big((p^{k-1}(p-1))\cdot 2\big) - (p-1)\big(4 + (p^{k-1}-1-1)\cdot 2\big) = 0.$$

Hence $d_g = 0$, for all $g \in \ker \epsilon_1$.

Since we have assumed $p \geq 3$, then we may have only p equal either to three or four. Furthermore, (4.3) gives for $p \geq 3$ the constriction $k \leq 2$.

We use (2.4) to obtain

$$\frac{p^{k-1}(p-1)}{2}\sum_g d_g = 4 + 0 + 2\cdot(p^k-2) = 2p^k \implies \sum_g (p-1)d_g = 4p.$$

However, $d_h = 0$ for all $g \in \ker \epsilon_1$, hence we can say

$$\sum_{g\notin \ker \epsilon_1} (p-1)d_g = \sum_g (p-1)d_g = 4p.$$

By the other side, we have also

$$4p = pl_{\epsilon_1} = \sum_{g\notin \ker \epsilon_1} \epsilon_1(g)d_g$$

hence

$$\sum_{g\notin \ker \epsilon_1} \epsilon_1(g)d_g = 4p = \sum_{g\notin \ker \epsilon_1} (p-1)d_g \implies \sum_{g\notin \ker \epsilon_1} (p-1-\epsilon_1(g))d_g = 0.$$

This gives $d_g = 0$ for each g having $\epsilon_1(g) \neq p-1$, while

$$d_g = \sum_{h\in\langle g\rangle} d_h = \frac{4p^{2-k}}{p-1} \qquad \text{for} \qquad \epsilon_1(g) = p-1.$$

Now it is useful to notice that $\frac{4p^{2-k}}{p-1}$ is an integer only for $p \in \{2,3,5\}$. Since we have assumed $p \geq 3$, then we may have only p equal either to three or five. For $p = 3$ we obtain the two families of the statement. We notice that from (3.2) both the families have canonical system base point free and equal to $|\pi^*\mathcal{O}_{\mathbb{P}^2}(1)|$.

We have only to discuss the remain families for $p = 5$. These families are excluded since any surface X of the family have geometric genus more than 3. Indeed, we have

$$5l_{2\epsilon_1} = \sum_g \epsilon_1(2g)d_g = \sum_g 3d_g = 3\sum_g d_g = 3\cdot\frac{4p}{p-1} = 3\cdot 5.$$

Thus, $l_{2\epsilon_1} = 3$, so that $p_g(X) \geq 4$. □

It remains to discuss only the $(k,p)-$planes of "type B". In order to do this, we first give some preliminary results.

Lemma 4.7. *Any generic (k,p)-plane satisfies the condition*

$$G = \langle g \mid d_g \neq 0\rangle.$$

In particular, we have

$$k \leq |\{g \in G | d_g \neq 0\}|.$$

Proof. The set of elements $g \in G$ such that $d_g \neq 0$ contains a set of generators for G. Indeed, if by contradiction $G \neq \langle g \colon d_g \neq 0\rangle$ then we can select a character $\chi \neq 0$ vanishing on $\langle g \colon d_g \neq 0\rangle$ and then the Pardini equation 2.4 would give $l_\chi = 0$. This contradicts the fact X is connected.

Furthermore, in our case $G \cong \mathbb{Z}_p^k$, and so the cardinality of a minimal set of generators of G equals the exponent k. □

Lemma 4.8. *There exists a family of smooth (k,p)-planes X satisfying the condition*

$$k+1 = |\{g \in G | d_g \neq 0\}| = \sum_g d_g$$

if and only if $k \geq 2$.
Moreover, this family is unique up to isomorphisms of G, and it is defined by choosing the building data as follows

$$d_{e_1} = \dots = d_{e_k} = d_{(p-1)\sum_k e_k} = 1, \qquad \textit{and} \qquad d_g = 0 \quad \textit{otherwise.}$$

In particular, the geometric genus of X equals

$$p_g(X) = \frac{1}{12}p^{k-2}\left(\frac{(k-2)(3k-5)}{2}p^2 - 3(k+1)(k-2)p + \frac{(k+1)(3k+2)}{2}\right) - 1.$$

Remark 4.9. The constructed family is a particular case of surfaces already studied by Hirzebruch in [5]. Moreover, using formulas [5, p. 124] for K_X^2 and $e(X)$, then one can compute the geometric genus of X.

We point out that a proof on the existence and uniqueness of the family can be given also by applying [2, Lemma 3.8 and Theorem 3.11].

However, we think that it can be useful for the comprehension of the rest of the section to give a short proof of this lemma.

Proof. Up to isomorphisms of G, we have

$$d_{e_1} = d_{e_2} = \dots = d_{e_k} = d_g = 1 \qquad \text{and} \qquad d_h = 0 \qquad \text{otherwise} \tag{4.5}$$

for a suitable $g \in G$. It is easy to determine the unique $g \in G$ for which Pardini equations 2.4 are satisfied:

$$pl_{\epsilon_j} = 1 + \epsilon_j(g) \leq 1 + (p-1) = p \implies l_{\epsilon_j} = 1 \text{ and } \epsilon_j(g) = p-1.$$

We have obtained

$$g = \sum_j \epsilon_j(g)e_j = (p-1)\sum_j e_j.$$

The case $k=1$ is excluded from the smoothness of X since we would have $d_{e_1} = d_{(p-1)e_1} = 1$ and this would contradict (4.1).

It remains to compute the geometric genus of X. Since X is a Galois cover of $\mathbb{P}^2$, then it has irregularity zero, from the Remark 2.11. This together with Noether formula give

$$p_g(X) = \chi(\mathcal{O}_X) - 1 = \frac{1}{12}(K_X^2 + e(X)) - 1.$$

Hence we just need to compute K_X^2 and $e(X)$.

Firstly, we determine K_X^2 using [8, Proposition 4.2]:

$$K_X^2 = p^k \left(-3 + \frac{(p-1)}{p}(k+1)\right)^2 = p^{k-2}\left((k-2)p - (k+1)\right)^2.$$

We determine $e(X)$ by applying the formula of Proposition 2.10:

$$e(X) = p^{k-2}\left(\frac{(k-1)(k-2)}{2}p^2 - (k+1)(k-2)p + \frac{k(k+1)}{2}\right).$$

□

We have a stronger condition for (k,p)-planes of "type B".

Proposition 4.10. *Any generic (k,p)-plane of "type B" satisfies the condition*

$$k+2 \leq \sum_g d_g.$$

Proof. From the Lemma 4.7 we get

$$k \leq |\{g \in G | d_g \neq 0\}| \leq \sum_g d_g.$$

The case $k = \sum_g d_g$ is not possible otherwise there would exist exactly k elements of G with $d_g \neq 0$ and that are generating G, so they are linearly independent. Moreover, all of them have $d_g = 1$. Then, up to isomorphisms of G, we say

$$d_{e_1} = d_{e_2} = \cdots = d_{e_k} = 1, \qquad \text{and} \qquad d_g = 0 \quad \text{otherwise}.$$

This gives a contradiction since

$$pl_{\epsilon_1} = d_{e_1} = 1.$$

It remains to exclude the case $k+1 = \sum_g d_g$. We have to distinguish

$$|\{g \in G | d_g \neq 0\}| = k \qquad \text{or} \qquad |\{g \in G | d_g \neq 0\}| = k+1.$$

The first case is not possible since up to isomorphisms of G

$$d_{e_1} = d_{e_2} = \ldots d_{e_{k-1}} = 1, \qquad d_{e_k} = 2 \qquad \text{and} \qquad d_g = 0 \text{ otherwise}.$$

In this case, we would have $pl_{\epsilon_1} = 1$.

Instead, in the second case we fall in the hypothesis of the Lemma 4.8. Hence firstly we have $k \geq 2$. If $k=2$, then the geometric genus of X equals $p_g(X) = \frac{(k+1)(3k+2)}{24} - 1 = 0$, which is not admissible.

Instead, if $k \geq 3$, then is straightforward to see $p_g(X) \geq 4$ by using the formula of Lemma 4.8. □

Notation 4.11. Given a Galois covering of $\mathbb{P}^2$, we denote by l the number of $\chi \in G^*$ such that $l_\chi = 1$.

Proposition 4.12. *The generic (k,p)-planes of "type B" satisfy the condition*

$$\sum_g d_g = 4 + \frac{2(2p^{k-1} + 1 - l)}{p^{k-1}(p-1)}.$$

Proof. We apply the Proposition 2.4 for $G \cong \mathbb{Z}_p^k$ to obtain

$$\frac{p^{k-1}(p-1)}{2} \sum_g d_g = \sum_\chi l_\chi = 3 \cdot 3 + 0 + 2(p^k - (l+4)) + l = 2p^k + 1 - l.$$

Hence

$$\sum_g d_g = \frac{2(2p^k + 1 - l)}{p^{k-1}(p-1)} = 4 + \frac{2(2p^{k-1} + 1 - l)}{p^{k-1}(p-1)}.$$

□

Corollary 4.13. *Smooth $(1,p)$-planes of "type B" do not exist. In other words, $k \geq 2$.*

Proof. The case $p = 3$ is obvious, since $k = 1$ and $G \cong \mathbb{Z}_3$ is a group of cardinality 3 that can not satisfy the condition to being of "type B".

Hence we suppose $p \geq 5$. Since G is a cyclic group and the $(1,p)$-plane is smooth, so the condition (4.1) has to be satisfied, then there is at most one je_1 has $d_{je_1} \neq 0$. This implies $d_{je_1} = \sum_g d_g = 4 + \frac{2(3-l)}{(p-1)}$ from the Proposition 4.12. We note that for $p \geq 7$ then

$$d_{je_1} = 4 + \frac{2(3-l)}{p-1} \leq 4 + \frac{6}{p-1} \leq 5 < p.$$

However, we would have

$$pl_{\epsilon_1} = jd_{je_1} \implies p \mid d_{je_j},$$

that is a contradiction.

For $p = 5$, then $0 \leq l \leq p - 4 = 1$ and so $d_{je_1} = 4 + \frac{3-l}{2}$ is an integer only for $l = 1$. In this case, then $d_{je_1} = 5$ and $l_{\epsilon_1} = j$ implies $j = 1$ or $j = 3$. If $j = 1$, then $l_{4\epsilon_1} = 4$, which is not possible, while if $j = 3$, then $l_{3\epsilon_1} = 4$, that is not possible too. In other words, the case $p = 5$ does not occur. □

Theorem 4.14. *Any generic $(k,p)-$plane of "type B" falls in one of the following cases*

a1) $p = 3$ *and* $k \in \{2,3\}$, $\sum_g d_g = 5$, $l = 3^{k-1} + 1$;
a2) $p = 3$ *and* $k \in \{2,3,4\}$, $\sum_g d_g = 6$, $l = 1$;
b) $p = 5$, *and* $k \in \{2,3\}$, $\sum_g d_g = 5$, $l = 1$;
c) $k = 2$, $\sum_g d_g = 4$, $l = 2p + 1$;

Proof. From the propositions 4.10 and 4.12 we get

$$k+2 \leq \sum_g d_g = 4 + \frac{2(2p^{k-1}+1-l)}{p^{k-1}(p-1)} \tag{4.6}$$

which implies for $l = 2p^{k-1}+1$ that $k \leq 2$ and $\sum_g d_g = 4$. Moreover, $k = 2$ from the Corollary 4.13.

Furthermore, for $p \geq 7$ there are no other possibilities than $k = 2$ and $\sum_g d_g = 4$. Indeed from the Corollary 4.13 we have $k \geq 2$ and so (4.6) gives

$$4 \leq k+2 \leq 4 + \frac{2(2p^{k-1}+1-l)}{p^{k-1}(p-1)} \implies \frac{2(2p^{k-1}+1-l)}{p^{k-1}(p-1)} \geq 0$$

so $\frac{2(2p^{k-1}+1-l)}{p^{k-1}(p-1)}$ is a non negative integer. However, we observe

$$\frac{2(2p^{k-1}+1-l)}{p^{k-1}(p-1)} \leq \frac{4}{p-1} + \frac{2}{p^{k-1}(p-1)} \tag{4.7}$$

which is less than 1 for $p \geq 7$. This forces $\frac{2(2p^{k-1}+1-l)}{p^{k-1}(p-1)}$ being equal to zero, so that $l = 2p^{k-1}+1$, $k = 2$, $\sum_g d_g = 4$.

Instead, for $p = 3, 5$ more cases may occur. For $p = 5$ then (4.7) tells us another possibility is $\frac{2(2p^{k-1}+1-l)}{p^{k-1}(p-1)}$ equal to 1, that it happens for $l = 1$. In this case, then (4.6) gives $k \leq 3$, $\sum_g d_g = 5$.

Finally, for $p = 3$ then (4.7) tells us other two possibilities are $\frac{2(2p^{k-1}+1-l)}{p^{k-1}(p-1)}$ either equal to 1 or 2. In the first case, we have $l = 3^{k-1}+1$, $k \leq 3$, $\sum_g d_g = 5$, whilst in the second case we have $l = 1$, $k \leq 4$, $\sum_g d_g = 6$. □

Notation 4.15. We fix a notation similar to [4, Not. 3.9]. The weight $(w_1(g), w_2(g))$ of an element $g = (g_1, \dots, g_k) \in \mathbb{Z}_p^k$ is a pair where $w_1(g)$ is the number of g_i different from zero, and $w_2(g)$ is the sum of the entries g_i of g. For instance, given $k = p = 3$, then the weight of $g = (1,2,2)$ is $(3,5)$.

We apply this notation to both the elements of G and G^*.

Before to discuss any case of Theorem 4.14 we give the following useful remark, for $p = 3$.

Remark 4.16. Assume that $p = 3$. Given χ such that $l_\chi = 1$, then $l_{2\chi} \in \{1,2\}$. Indeed, from 2.4 we have

$$3 = 3l_\chi = \sum_g \chi(g) d_g.$$

The number 3 can be written only either as $1+1+1$ or $1+2$ or 3, so either it there exists g_1, g_2, g_3 such that $\chi(g_i) = d_{g_i} = 1$ for any i or it there exists g_1, g_2 such that $\chi(g_1) = d_{g_1} = 1$ and $\chi(g_2) d_{g_2} = 2$, or there is only one g with $\chi(g) = 1$ and $d_g = 3$.

In the first and third case we would have $l_{2\chi} = 2$, whilst in the second case $l_{2\chi} \in \{1,2\}$.

Proposition 4.17. *The generic (k,p)-planes of the case a1) of Theorem 4.14 form one family, with $k=p=3$. These surfaces have a canonical system that is base point free and*

$$l_\chi = 3 \text{ if } w_1(\chi)=1, \text{ and } w_2(\chi)=1 \qquad l_\chi = 1 \text{ if } w_1(\chi)=2 \text{ and } w_2(\chi)\leq 3 \text{ or } \chi = 2(\epsilon_1+\epsilon_2+\epsilon_3) \qquad l_\chi = 2 \text{ otherwise;}$$

$$d_g = 1 \text{ if } w_1(g)=3, \text{ and } w_2(g)=5 \qquad d_g = 2 \text{ if } g = 2(e_1+e_2+e_3), \qquad d_g = 0 \text{ otherwise.}$$

Proof. (k,p)-planes of "type B" have three character χ_1,χ_2,χ_3 with $l_{\chi_i}=3$. This forces $l_{2\chi_i}=2$ for any $i=1,2,3$. Indeed, the Equation 2.1 applied to $\chi=\chi_i$ and $\chi'=2\chi_i$ gives

$$3+l_{2\chi_i} = l_{\chi_i}+l_{2\chi_i} = 0+\sum_{g\notin\ker\chi_i} d_g \leq \sum_g d_g = 5 \implies l_{2\chi_i}\leq 2. \tag{4.8}$$

However, $l_{2\chi_i}$ equal to 1 is excluded since otherwise $l_{\chi_i}=l_{2(2\chi_i)}$ would be either 1 or 2 from the Remark 4.16.

We have then one element 0 with $l_0=0$, $3^{k-1}+1$ elements χ having $l_\chi=1$, at least three $2\chi_i$ having $l_{2\chi_i}=2$, and others three χ_i having $l_{\chi_i}=3$. In total they are $1+3^{k-1}+1+3+3$, which is either 11 or 17. This excludes $k=2$ since $G\cong\mathbb{Z}_3^2$ consists of only nine elements.

Hence we assume $k=3$. From Lemma 4.7 then we need at least three elements g_1,g_2,g_3 having $d_{g_i}\neq 0$.

We claim that the number of elements g with $d_g\neq 0$ is not three. Indeed, if there are exactly three elements g_1,g_2,g_3 with $d_{g_i}\neq 0$, then they form a basis of $G\cong\mathbb{Z}_3^3$.
Up to relabel g_i, we only may have either $d_{g_1}=d_{g_2}=1, d_{g_3}=3$ or $d_{g_1}=1, d_{g_2}=2, d_{g_3}=2$. This gives a contradiction since if η_1,η_2,η_3 is the dual basis of g_1,g_2,g_3, then $3l_{\eta_1}=d_{g_1}=1$.

Hence the number of elements g with $d_g\neq 0$ is at most four.

We observe that the Equation (4.8) implies $\sum_{g\notin\ker\chi_i} d_g=\sum_g d_g=5$, so that any $g\in G$ for which $d_g\neq 0$ does not vanish on χ_i, for any $i=1,2,3$.

However, if $\chi_3\in\langle\chi_1,\chi_2\rangle$, then there are exactly $2\cdot2\cdot3-2\cdot3=6$ elements that does not vanish on χ_i. We can select from this set at most 3 elements g_1,g_2,g_3 satisfying the smoothness condition (4.1) for which $\langle g_i\rangle\cap\langle g_j\rangle=\{0\}$ and such that $d_{g_i}\neq 0$. Hence the case $\chi_3\in\langle\chi_1,\chi_2\rangle$ has to be excluded.

Let us discuss the case χ_1,χ_2,χ_3 are linearly independent. Here the number of elements not vanishing on χ_i is $2^3=8$. Hence we can select from this set at most 4 elements $g_1,\dots,g_4$ satisfying $\langle g_i\rangle\cap\langle g_j\rangle=\{0\}$ and with $d_{g_i}\neq 0$. As already observed above, we can not choice less than four elements, so they are exactly four. Furthermore, up to relabel g_i, we only may have

$$d_{g_1}=d_{g_2}=d_{g_3}=1, \qquad \text{and} \qquad d_{g_4}=2.$$

Now we use the equation of Corollary 2.4 to obtain

$$6 = 3l_{2\chi_i} = \chi_i(2g_1) + \chi_i(2g_2) + \chi_i(2g_3) + 2\chi_i(2g_4).$$

This gives $\chi_i(2g_4) = 1$ and from the smoothness condition (4.1) we also deduce, up to relabel g_1, g_2, g_3, that $\chi_i(2g_i) = 2$, $\chi_i(2g_j) = 1$, $j \neq i$. In other words, if we fix the dual basis of χ_1, χ_2 and χ_3, then we get

$$g_1 = (1,2,2), \quad g_2 = (2,1,2), \quad g_3 = (2,2,1), \quad \text{and} \quad g_4 = (2,2,2).$$

By (3.2) the canonical system $|K_X|$ is generated by the following three divisors

$$R_{g_1}, \qquad R_{g_2}, \qquad \text{and} \qquad R_{g_3}$$

and then from the smoothness assumption the base locus is empty. In other words, $|K_X|$ is base point free. □

Remark 4.18. Let us consider a generic $(k,p)-$plane of type "B" having $l = 1$, and so falling in one of the cases $a2)$ with $p = 3$ or $b)$ with $p = 5$ of Theorem 4.14.

$l = 1$ means that there exists only one $\chi \in G^*$ such that $l_\chi = 1$. Let $\chi_1, \chi_2, \chi_3 \in G^*$ be the characters with $l_{\chi_i} = 3$. Let us define

$$\alpha := |\{\chi_1, \chi_2, \chi_3\} \cap \langle\chi\rangle|.$$

We apply Theorem 2.9 to $\mathcal{H} = \langle\chi\rangle$ to obtain

$$\begin{aligned}
&\frac{p^{k-1}(p-1)}{2} \sum_{g \in \ker \chi} d_g = \sum_{\eta \notin \langle\chi\rangle} l_\eta - (p^{k-1} - 1) \sum_{\eta \in \langle\chi\rangle} l_\eta \\
&= 3 \cdot (3-\alpha) + 2 \cdot (p^k - p - (3-\alpha)) - (p^{k-1} - 1) \cdot (1 + 3 \cdot \alpha + 2 \cdot (p - 2 - \alpha)) \\
&= (3-\alpha)p^{k-1}.
\end{aligned}$$

Thus

$$\sum_{g \in \ker \chi} d_g = \frac{2(3-\alpha)p^{k-1}}{p^{k-1}(p-1)} = \frac{2(3-\alpha)}{p-1}. \tag{4.9}$$

If p is equal to three, then we have $\alpha = 0$ and $\sum_{g \in \ker \chi} d_g = 3$. In fact, the only possible values of α are either 0 or 1, and $\alpha = 1$ is not possible since $l_\chi = 1$ implies $l_{2\chi} \in \{1, 2\}$, from the Remark 4.16.

Instead, if p is equal to five, then the values of α for which $\frac{2(3-\alpha)}{p-1}$ is an integer are either 1 or 3.

Proposition 4.19. *The generic (k,p)-planes of the case $a2)$ of Theorem 4.14 form one family, with $k = 2$ and $p = 3$. These surfaces have a canonical system that is not base point free and*

$$\begin{array}{lll}
l_\chi = 3 \textit{ if } \chi(e_2) = 1, & l_{\epsilon_1} = 1 & l_\chi = 2 \textit{ otherwise}; \\
d_g = 1 \textit{ if } \epsilon_1(g) = 1, & d_{2e_2} = 3 & d_g = 0 \textit{ otherwise}.
\end{array}$$

Proof. From the Remark 4.18 we have $\{\chi_1, \chi_2, \chi_3\} \cap \langle\chi\rangle = \emptyset$ and $\sum_{g\in\ker\chi} d_g = 3$.

Let $g \notin \ker\chi$ and denote by $\beta = \beta(g) \in \{0, 1, 2, 3\}$ the number of χ_i with $l_{\chi_i} = 3$ vanishing on g. Then Theorem 2.9 applied to $\mathcal{H} = \ker g$ gives

$$\begin{aligned}\frac{p^{k-1}(p-1)}{2}\sum_{h\in\langle g\rangle} d_h &= 1 + 3\cdot(3-\beta) + 2(p^k - p^{k-1} - 1 - (3-\beta))\\ &\quad - (p-1)(3\beta + 0 + 2(p^{k-1} - 1 - \beta))\\ &= 10 - 3\beta + 2p^{k-1}(p-1) - 8 + 2\beta - \beta(p-1)\\ &\quad + 2p^{k-1}(p-1) + 2(p-1)\\ &= (2-\beta)p\end{aligned}$$

that implies

$$\sum_{h\in\langle g\rangle} d_h = \frac{2(2-\beta)}{p-1}p^{2-k} = (2-\beta)3^{2-k}.$$

This excludes β equal to three. Furthermore, it has to exists at least one $g \notin \ker\chi$ such that $\beta = \beta(g) \neq 2$ otherwise we would have

$$6 = \sum_g d_g = \sum_{g\in\ker\chi} d_g + \sum_{g\notin\ker\chi} d_g = 3 + 0 = 3.$$

Hence it there exists $g \notin \ker\chi$ for which β is equal either to 0 or to 1. In any case, this forces $k = 2$.

Moreover, $k = 2$ tells us $\ker\chi$ is a cyclic group and since $\sum_{g\in\ker\chi} d_g = 3$, then from the smoothness condition (4.1) we may only have one $g_1 \in \ker\chi$ for which $d_{g_1} = 3$.

From Lemma 4.7, then the number of g with $d_g \neq 0$ is at least 2. Furthermore, the numbers of lines of $G \cong \mathbb{Z}_3^2$ is four, so from the smoothness condition (4.1) we can select at most four g_i with $d_{g_i} \neq 0$. In other words, we are saying that

$$|\{g\colon d_g \neq 0\}| \in \{2, 3, 4\}.$$

The case equal to 2 is not possible since $3 + d_{g_2} = d_{g_1} + d_{g_2} = \sum_g d_g = 6$ and so $d_{g_2} = 3$. However, g_1 and g_2 form a basis of G and so if η_1 and η_2 is the dual basis, then we would get

$$3l_{\eta_i} = d_{g_i} = 3 \implies l_{\eta_i} = 1$$

that contradicts the assumption $l = 1$.

The case equal to 3 means that there are g_2 and g_3 such that $d_{g_2} = 1$ and $d_{g_3} = 2$, both not vanishing on χ. Then

$$3 = 3l_\chi = \chi(g_2) + \chi(g_3)2$$

which forces $\chi(g_2) = \chi(g_3) = 1$. Instead, we have

$$9 = 3l_{\chi_1} = \chi_1(g_1)3 + \chi_1(g_2) + \chi_1(g_3)2.$$

which forces $\chi_1(g_2) = \chi_1(g_3)$.

However, we have already observed χ_1 is not a multiple of χ, hence χ_1 and χ form a basis of G^*. Then $\chi(g_2) = \chi(g_3)$ and $\chi_1(g_2) = \chi_1(g_3)$ implies $g_2 = g_3$, which does not provide a smooth (k,p)-plane.

It remains the case for which there are g_2, g_3, g_4 with $d_{g_i} \neq 0$ all not vanishing on χ. They all have $d_{g_i} = 1$. Then

$$3 = 3l_\chi = \chi(g_1) + \chi(g_2) + \chi(g_3) \implies \chi(g_i) = 1.$$

However, since χ and χ_1 form a basis of G^* and we are looking for (k,p)-planes which are smooth, namely when (4.1) is satisfied, then we may only have in the dual basis on G

$$g_2 = (1,0), \qquad g_3 = (1,1), \qquad \text{and} \qquad g_4 = (1,2).$$

Finally, we use

$$9 = 3l_{\chi_1} = \chi_1(g_1)3 + 3 \implies \chi_1(g_1) = 2$$

to deduce $g_1 = (0,2)$.

By (3.2) the canonical system $|K_X|$ is generated by the following three divisors

$$2R_{g_2} + R_{g_3}, \qquad R_{g_2} + 2R_{g_4} \qquad \text{and,} \qquad 2R_{g_3} + R_{g_4}$$

and then by the smoothness assumption the base locus is the schematic intersection

$$R_{g_2} \cap R_{g_3}, \qquad R_{g_2} \cap R_{g_4}, \qquad \text{and} \qquad R_{g_3} \cap R_{g_4}.$$

The lines $D_{g_2}, D_{g_3}, D_{g_4}$ intersect pairwise transversally in one point. Above each of the intersection point $D_{g_i} \cap D_{g_j}$ there is only one point of X, stabilized by the entire group $\langle g_i, g_j\rangle = G$, the intersection point $R_{g_i} \cap R_{g_j}$, for any $i,j \in \{2,3,4\}$, $i \neq j$. A straightforward local computation shows that R_{g_i} and R_{g_j} are transversal. □

The rest of this section aims to prove that the remain cases of Theorem 4.14 do not provide others smooth (k,p)-planes of "type B".

Proposition 4.20. *The case b) of the Theorem 4.14 does not occur.*

Proof. The Remark 4.18 gives

$$\{\chi_1, \chi_2, \chi_3\} \subseteq \langle\chi\rangle \qquad \text{or} \qquad |\{\chi_1, \chi_2, \chi_3\} \cap \langle\chi\rangle| = 1,$$

so we have to distinguish these two cases.
In the first case, then $\sum_{h \in \ker\chi} d_h = 0$, from the Remark 4.18.
Let us consider $g \notin \ker\chi$. We apply Theorem 2.9 to $\mathcal{H} = \ker g$ to obtain

$$\begin{aligned}\frac{p^{k-1}(p-1)}{2} \sum_{h\in\langle g\rangle} d_h &= 1 + 3\cdot 3 + 2\cdot(p^k - p^{k-1} - 4) - (p-1)\cdot(0 + 2\cdot(p^{k-1}-1)) \\ &= 10 - 8 + 2p^{k-1}(p-1) - 2p^{k-1}(p-1) + 2p - 2 \\ &= 2p.\end{aligned}$$

Thus

$$\sum_{h\in\langle g\rangle} d_h = \frac{4p^{2-k}}{p-1} = 5^{2-k}.$$

This forces $k = 2$, $\sum_{h\in\langle g\rangle} d_h = 1$. We note that for $k = 2$ the complement of the kernel of χ consists of 20 elements and so we can select from it at most 5 elements satisfying the smoothness condition (4.1). However, $\sum_g d_g = 5$, and each selected element would give a contribution to the sum equal to 1.
This forces to select exactly five elements $g_1, \dots, g_5$ all not vanishing on χ, having $d_{g_i} = 1$, and satisfying (4.1).

Furthermore, we have

$$5 = 5l_\chi = \sum_{i=1}^{5} \chi(g_i)d_{g_i} = \sum_{i=1}^{5} \chi(g_i) \implies \chi(g_i) = 1, \ i = 1, \dots 5.$$

From the condition (4.1), in the dual basis of a completion to base of χ, we would have

$$g_1 = (1, 0), \quad g_2 = (1, 1) \quad g_3 = (1, 2), \quad g_4 = (1, 3), \quad g_5 = (1, 4).$$

However, this gives

$$5l_{j\chi} = 5 \cdot j \implies l_{j\chi} = j,$$

that is a contradiction since $l_{3\chi} = 3$ and $l_{4\chi} = 4$. This would give a (k, p)-plane not of geometric genus three, and so not of "type B".

Let us consider now the case where there is only one $\chi_i \in \langle\chi\rangle$, let us say χ_3. In this case, (4.9) becomes $\sum_{g\in\ker\chi} d_g = 1$.

We prove

$$\ker^c \chi \cap \ker \chi_1 \subseteq \ker \chi_2. \tag{4.10}$$

Let us choose $g \in \ker^c \chi \cap \ker \chi_1$ and suppose $g \notin \ker \chi_2$. Then we apply Theorem 2.9 to $\mathcal{H} = \ker g$ to obtain

$$\begin{aligned}\frac{p^{k-1}(p-1)}{2} \sum_{h\in\langle g\rangle} d_h &= 1 + 3 + 3 + 2(p^k - p^{k-1} - 3) - (p-1)(0 + 3 + 2(p^{k-1} - 2)) \\ &= 1 + 2p^{k-1}(p-1) + p - 1 - 2p^{k-1}(p-1) \\ &= p.\end{aligned}$$

Hence

$$\sum_{h\in\langle g\rangle} d_h = \frac{2p^{2-k}}{p-1} = \frac{5^{2-k}}{2}$$

that is not possible. Hence (4.10) follows.

Let us take now $g \in \ker^c \chi \cap \ker^c \chi_2$. Then $g \notin \ker \chi_1$ from (4.10), and we have

$$\begin{aligned}\frac{p^{k-1}(p-1)}{2} \sum_{h\in\langle g\rangle} d_h &= 1 + 3 + 3 + 3 + 2(p^k - p^{k-1} - 4) - (p-1)(0 + 2(p^{k-1} - 1)) \\ &= 2 + 2p^{k-1}(p-1) + 2p - 2 - 2p^{k-1}(p-1) \\ &= 2p.\end{aligned}$$

Thus

$$\sum_{h\in\langle g\rangle} d_h = p^{2-k} = 5^{2-k} = 1$$

that forces $k=2$. Then $k=2$ implies (4.10) means $\ker\chi_1=\ker\chi_2$, or in other words $\chi_2\in\langle\chi_1\rangle$.

Let us choose now $g\in\ker^c\chi\cap\ker\chi_1$. Then applying Theorem 2.9 to $\mathcal{H}=\ker g$ we obtain easily

$$\begin{aligned}\frac{p^{k-1}(p-1)}{2}\sum_{h\in\langle g\rangle} d_h &= 1+3+2(p^k-p^{k-1}-2)-(p-1)(3+3+0+2(p^{k-1}-3))\\ &= 2p^{k-1}(p-1)-2p^{k-1}(p-1)\\ &= 0.\end{aligned}$$

Then

$$\begin{aligned}5=\sum_g d_g = \sum_{g\in\ker\chi} d_g+\sum_{g\in\ker^c\chi\cap\ker\chi_1} d_g+\sum_{g\in\ker^c\chi\cap\ker^c\chi_1} d_g = 1+\sum_{g\in\ker^c\chi\cap\ker^c\chi_1} d_g\\ \implies \sum_{g\in\ker^c\chi\cap\ker^c\chi_1} d_g=4.\end{aligned}$$

This means there should be exactly four elements $g_1,\ldots,g_4$ all not vanishing on χ and χ_1 such that $d_{g_i}=1$. Let us denote by $g_5\in\ker\chi$ the unique element such that $d_{g_5}=1$.

Furthermore, $k=2$ implies that the cardinality of $\ker^c\chi\cap\ker\chi_1$ is $p^{k-1}-p^{k-2}=5-1=4$, and so this set equals $\ker\chi_1\setminus\{0\}$. Then (4.10) permits to conclude $\ker\chi_1=\ker\chi_2$, and so that $\chi_2\in\langle\chi_1\rangle$.

Now we observe

$$5=5l_\chi=\chi(g_1)+\cdots+\chi(g_4)+0$$

and so, up to relabel $g_1,\ldots,g_4$, we have $\chi(g_1)=\chi(g_2)=\chi(g_3)=1$, and $\chi(g_4)=2$. This gives also

$$5l_{j\chi}=3j+[2j]_5$$

so that $\chi_3=4\chi$. Furthermore, since χ and χ_1 are linearly independent and $k=2$ then they generate G^* and so from the smmoothness condition (4.1) we have $\chi_1(g_1)$,$\chi_1(g_2)$ and $\chi_1(g_3)$ are are pairwise distinct. Then up to isomorphisms of G we can assume

$$\chi_1(g_1)=2,\qquad \chi_1(g_2)=3\qquad\text{and}\qquad \chi_1(g_3)=4.$$

Moreover, this forces $\chi_1(g_4)=2$ since g_4 is chosen to being not a multiple of each one among g_1,g_2,g_3.

We can now compute $\chi_1(g_5)$, which is

$$15=5l_{\chi_1}=2+3+4+2+\chi_1(g_5)\implies\chi_1(g_5)=4.$$

In other words, fixing the dual basis of χ,χ_1 on G, then the only solution up to isomorphisms of G is given by

$$g_1=(1,2),\qquad g_2=(1,3),\qquad g_3=(1,4)\qquad g_4=(2,2),\qquad\text{and}\qquad g_5=(0,4).$$

However,

$$5l_{\chi+\chi_1}=3+4+0+4+4\implies l_{\chi+\chi_1}=3$$

and so we would have $l_{4\chi} = l_{\chi_1} = l_{2\chi_1} = l_{\chi+\chi_1} = 3$. This means that the $(k,p)-$plane constructed has not geometric genus three, and so is not of "type B". □

Proposition 4.21. *The case c) of the Theorem 4.14 does not occur.*

Proof. The case $p = 3$ is obviously excluded since the group $G = \mathbb{Z}_3^2$ consists of 9 elements and we are requiring that $2p + 1 = 7$ elements have $l_\chi = 1$, three have $l_\chi = 3$, and $l_0 = 0$.

We are going to construct all $(2,p)-$planes with $\sum_g d_g = 4$ and then prove that no one of these have geometric genus three, and so they are not of "type B".

Since $\sum_g d_g = 4$, then from Lemma 4.7 there are four possibilities, up to isomorphisms of G:

$$d_{e_1} = 2, d_{e_2} = 2, \quad \text{or} \quad d_{e_1} = 1, d_{e_2} = 3, \quad \text{or} \quad d_{e_1} = d_{e_2} = 1, d_g = 2,$$
$$\text{or} \quad d_{e_1} = d_{e_2} = d_{g_1} = d_{g_2} = 1.$$

The first two cases are easily not admissible because we would have $pl_{\epsilon_1} = d_{e_1}$. The third case is more interesting:

$$pl_{\epsilon_j} = 1 + \epsilon_j(g)2 \leq 1 + 2(p-1) = 2p - 1 \implies l_{\epsilon_j} = 1 \quad \text{and} \quad \epsilon_j(g) = \frac{p-1}{2}.$$

Thus $g = \sum_j \epsilon_j(g)e_j = \frac{p-1}{2}(e_1 + e_2)$. However, for any α such that $1 \leq \alpha \leq \frac{p-3}{2}$ we have

$$\begin{aligned} pl_{(p-2\alpha)\epsilon_1+(p-1)\epsilon_2} = pl_{(p-1)\epsilon_1+(p-2\alpha)\epsilon_2} &= (p-2\alpha) + (p-1) + [\frac{p-1}{2}(2p-(2\alpha+1))]_p 2 \\ &= 2p - (2\alpha+1) + [\alpha + \frac{p+1}{2}]_p 2 \\ &= 2p - (2\alpha+1) + 2\alpha + p + 1 = 3p. \end{aligned}$$

This implies $l_{(p-2\alpha)\epsilon_1+(p-1)\epsilon_2} = l_{(p-1)\epsilon_1+(p-2\alpha)\epsilon_2} = 3$. In other words, each value of α gives two distinct $\chi_1, \chi_2 \in G^*$ for which $l_{\chi_1} = l_{\chi_2} = 3$. However, when $p \geq 7$, we can always assume either α equals to 1 or 2, which implies $p_g(X) \geq 4$. Instead, for $p = 5$, we easily compute $p_g(X) = 2$.

It remains the last fourth case. We can observe that

$$\begin{aligned} pl_{\epsilon_j} = 1 + \epsilon_j(g_1) + \epsilon_j(g_2) \leq 1 + (p-1) + (p-1) = 2p - 1 \\ \implies l_{\epsilon_j} = 1 \quad \text{and} \quad \epsilon_j(g_2) = p - 1 - \epsilon_j(g_1). \end{aligned}$$

Hence $g_2 = (p-1)(e_1 + e_2) - g_1$. In other words, for any choice of g_1 we get a distinct family of $(2,p)$-planes. We compute the geometric genus of each of these $(2,p)$-planes and we show that they are not equal to three.

From the Remark 2.11 we know that they have irregularity zero. Hence using Noether formula we get

$$p_g(X) = \frac{1}{12}(K_X^2 + e(X)) - 1.$$

The self-intersection of a canonical divisor is computable via [8, Proposition 4.2]:

$$K_S^2 = p^2 \left(-3 + \frac{p-1}{p} 4\right)^2 = (p-4)^2.$$

Instead, to determine $e(X)$ we use the formula of Proposition 2.10:

$$\begin{aligned} e(X) &= p^2 \left(3 - 4 \cdot 2 \cdot \frac{p-1}{p} + 6 \frac{(p-1)^2}{p^2}\right) \\ &= 3p^2 - 8p^2 + 8p + 6p^2 - 12p + 6 \\ &= p^2 - 4p + 6. \end{aligned}$$

This means

$$p_g(X) = \frac{1}{6}(p-1)(p-5).$$

However, it is straightforward to see that no value of p may attain $p_g(X)$ equal to three. □

We can finally state and prove the main theorem of this note.

Theorem 4.22. *Let $p \geq 3$ be a prime number. All smooth (k, p)-covers X of the plane with geometric genus 3 have $p = 3$ and are regular surfaces with ample canonical class.*

They form four unirational families, that we labeled as A1, A2, B3, C2, in a way that A1 is a family of triple planes, A2 and C2 are two families of bi-triple planes and B3 is a family of $(3, 3)$-planes.

The degree of the canonical map φ_{K_X} is constant in each family.

We summarize in the following table the modular dimension (the dimension of its image in the Gieseker moduli space of the surfaces of general type) of each family, and the values of K_X^2 and $\deg \varphi_{K_X}$ of each surface in the family: The

Family	*A1*	*A2*	*B3*	*C2*
mod. dim.	*19*	*7*	*3*	*7*
K_X^2	*3*	*9*	*3*	*9*
$\deg \varphi_{K_X}$	*3*	*9*	*3*	*3*

canonical map is a morphism of degree K_X^2 on $\mathbb{P}^2$ unless X of type $C2$, in which case the canonical map is a rational map of degree $K_X^2 - 3 \cdot 2 = 3$ undefined at 3 points.

Proof. From the Remark 2.11, each surface X that is a Galois cover $\pi \colon X \to \mathbb{P}^2$ has irregularity zero.

The value of the self-intersection of the canonical class follows by the formula (see [8, (4.8)])

$$K^2 = 3^{k-2} \left(-9 + 2 \sum_{g \in G} d_g\right)^2.$$

By Propositions 4.6, 4.17, 4.19, the canonical system of X is base point free unless X is of type C2, in which case it has three base points with schematic multiplicity 2. So (blowing up the base points in this last case) we get a surface with canonical system having movable part of self intersection as in the third line of the table above, so strictly positive. Then the canonical map is not composed with a pencil. Since $p_g = 3$ then the canonical map of this surface is a morphism on $\mathbb{P}^2$ of the given degree.

The families are parametrized by a Zariski open subset of a product of projective spaces, the complete linear systems to which the divisors $|D_g|$, quoted by the faithful action of PGL(3), a group of dimension 8. Since the surfaces are of general type, their automorphism group is finite and therefore it contains only finitely many subgroups of the form $(\mathbb{Z}/3\mathbb{Z})^k$, which implies that the map from this quotient to the Gieseker moduli space of the surfaces of general type is finite. So the modular dimension of each family equals

$$-8 + \sum \dim |D_g|$$

which gives the modular dimensions in the table above. As an example, the family $C2$ depends on the choice of three lines, and a cubic so its modular dimension is

$$-8 + 3 \cdot 2 + 9 = 7.$$

□

References

[1] Bauer, I., Pignatelli, R.: Rigid but not infinitesimally rigid compact complex manifolds, Duke Math. J. **170**, 1757–1780, (2021); doi:10.1215/00127094-2020-0062

[2] Böhning, C., von Bothmer, HC.G., Pignatelli, R., A rigid, not infinitesimally rigid surface with K ample. Boll Unione Mat Ital 15, 57–85 (2022); https://doi.org/10.1007/s40574-021-00296-3

[3] Du, R., Gao, Y.: Canonical maps of surfaces defined by abelian covers, Asian J. Math., **18**, 219–228, (2014); doi:10.4310/AJM.2014.v18.n2.a2

[4] Fallucca F., Pignatelli R., Smooth k-double covers of the plane of geometric genus three, Rend. Mat. Appl., **45**, 153–180, (2024),

[5] Hirzebruch F., Arrangements of Lines and Algebraic Surfaces, Artin, M., Tate, J. (eds) Arithmetic and Geometry, Progress in Mathematics, **36**, (1983)

[6] Liedtke, C.: Singular abelian covers of algebraic surfaces, Manuscripta Math. **112**, 375–390, (2003), doi:10.1007/s00229-003-0408-y

[7] Reid, M.: Young person's guide to canonical singularities, Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, **46**, 345–414, (1987)

[8] Pardini, R.: Abelian covers of algebraic varieties, J. Reine Angew. Math. **417**, 191–213, (1991); doi:10.1515/crll.1991.417.191

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